# SYSTEMATIC SEARCHING FOR CAVITY AND WAVEGUIDE MODES BY MAKING USE OF FOSTER'S THEOREM

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## ABSTRACT

Resonance and irrotational cavity modes as well as waveguide eigenmodes are usually determined numerically (except for a few structures whose solutions are analytically known). Most of the methods of analysis end up at a determinental equation, the zeros of which characterize these modes. Numerically overlooking some of these zeros is one of the problems which greatly degrade the accuracy of a field expansion in terms of the corresponding modes.

In this contribution it is shown that it is always possible to find a reactance (or susceptance) function having the same set of zeros as the original determinental equation. This enables making use of Foster's theorem to systematically determine these zeros.

## STATEMENT OF THE PROBLEM

Field expansion in terms of complete sets of modal functions is the spectral domain version of discretizing field problems making them suitable for numerical solutions. These modal sets must be a priori known and for most of the structures of practical importance have to be determined numerically. We can in general distinguish between two types of structures: Guiding structures for which a field expansion is made in terms of the corresponding propagation modes ([1]-[7]) and cavity resonators which are characterized by both the divergence-free resonance modes and the curl-free ones ([8]–[12]). Because waveguides can be considered two-dimensional resonators (applying, e.g., the transverse resonance technique [13]), only resonator problems will be considered here.

An arbitrarily shaped cavity resonator can always be treated as a cascaded connection of line sections, step discontinuities and tapers of waveguides (along a properly selected direction) which is short-circuited at two end planes. A spherical resonator, e.g., is a taper with a semi-circular profile of circular waveguides along any direction. Assuming that the eigenmodes of the corresponding waveguides are known, the generalized scattering matrix of the cascaded connection is determined (see, e.g., [9]). The number of waveguide modes used to expand the field in the different waveguide sections is chosen to have the same spatial resolution allover the structure. The size of the resulting scattering matrix is determined by the number of modes used to expand the field in the waveguide sections adjacent to the end planes. The end planes are next short-circuited resulting in a determinental equation, the zeros of which correspond to the resonance frequency of the different resonance modes. Irrotational modes are similarly determined as has been shown in [12]. The zeros of the above described determinental equation are in general very nonuniformly distributed along the frequency axis. They usually interlace the poles of the same determinant in an irregular fashion.

The numerical searching for the zeros of a function can be made by using either a constant or an adaptive searching step. The constant step must be smaller than the smallest difference between two adjacent zeros or adjacent zero and pole. Otherwise, the searching program will fail to localize those zeros which are near to other zeros or poles. Such a searching strategy is extremly inefficient. Furthermore it cannot be guaranteed that all zeros are detected because the smallest distance between adjacent zeros or adjacent zero and pole is mostly not known a priori and consequently may be smaller than the searching step (even if a very small one is used). The adaptive searching step on the other hand is continuously changed according to the characteristics of the function. Such a searching strategy is numerically very efficient saving a great deal of cpu time. But one has to keep in mind that even with a sophisticated control of the searching step some zeros may be overlooked.

Reactance and susceptance functions of frequency corresponding to lossless structures are governed by Foster's theorem [14]. They are increasing functions for all frequencies with their zeros and poles taking place alternately. These characteristics considerably facilitate the process of searching for the zeros of a reactance function which leads to a significant reduction in cpu time requirements resulting in improving the efficiency of the numerical algorithm. Furthermore it can be guaranteed that none of the zeros is overlooked.

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## BASIC FORMULATION

In order to explain the idea of this contribution we consider again the cascaded structure considered before at one of the short circuited end planes. Instead of determining the generalized scattering matrix, we will consider the corresponding generalized impedance matrix, which relates the generalized voltage vector $\boldsymbol{V}$ (representing the complex amplitudes of the modal transverse electric fields in the end line just before the short cicuit) to the generalized current vector $\boldsymbol{I}$ (representing the complex amplitudes of the corresponding modal transverse magnetic fields) by

$$\boldsymbol{V} = [Z]\,\boldsymbol{I} \quad . \tag{1}$$



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The conventional resonance condition is obtained by putting $V = 0$ (due to the short circuit) and looking for the nonvanishing solutions for $I$, which correspond to those frequencies (or wavenumbers) which singularize the generalized impedance matrix $[Z]$ according to the determinental equation

$$\mathrm{Det}\{[Z]\} = 0 \quad . \tag{2}$$

Because the quantity $\mathrm{Det}\{[Z]\}$ is not a reactance function, even if the structure is assumed lossless, Foster's theorem cannot be applied. In order to have a reactance function, we will first short-circuit all modes of the end line section except for a single one (say the $i$th one). The input impedance corresponding to the $i$th mode is next calculated.

$$V_i = Z_i^{(\mathrm{in})} I_i \tag{3}$$

It is readily proved that $Z_i^{(\mathrm{in})}$ is related to the $i$th diagonal element $Y_{ii}$ of the generalized admittance matrix $[Y] = [Z]^{-1}$ by

$$Z_i^{(\mathrm{in})} = \frac{1}{Y_{ii}} \quad . \tag{4}$$

For lossless structures, $Z_i^{(\mathrm{in})}$ is purely imaginary.

$$Z_i^{(\mathrm{in})} = jX_i^{(\mathrm{in})} \tag{5}$$

where $X_i^{(\mathrm{in})}$ is a reactance function of frequency (or wavenumber), which is governed by Foster's theorem. The zeros of $X_i^{(\mathrm{in})}$ are the looked for resonance frequencies (or resonance wavenumbers) of the structure.

For a prescribed spatial resolution, we retain a maximum number of modes $M$ in the field expansion corresponding to the end line section considered above. Because $i$ has been arbitrarily chosen, we will have a number $M$ of resonance frequency sets, each represents the zeros of $X_i^{(\mathrm{in})}$; $i = 1, 2, \ldots, M$. These sets must however be identical because these zeros correspond to short-circuiting all modes ($i = 1, 2, \ldots, M$). On the other hand, a pole of $X_i^{(\mathrm{in})}$ cannot be identical with a pole of $X_j^{(\mathrm{in})}$. The first pole corresponds to open-circuiting the $i$th mode and short-circuiting all other modes (including the $j$th one) while the second pole corresponds to open-circuiting the $j$th mode and short-circuiting all other modes (including the ith one).

Another essential difference between the two reactance functions $X_i^{(\mathrm{in})}$ and $X_j^{(\mathrm{in})}$ is that $X_i^{(\mathrm{in})}$ represents the structure viewed from the $i$th mode. If the $i$th mode is propagating and strongly excited in a certain frequency range, $X_i^{(\mathrm{in})}$ will telescope all details of the structure giving rise to a clear picture of it. In this case there will be a considerable difference between short- and open-circuiting the $i$th mode which results in a significant distance between a zero and a neighbouring pole belonging to this frequency range. On the other hand, if the $i$th mode is evanescent or weakly excited $X_i^{(\mathrm{in})}$ will give a very fuzzy picture of the structure. Short- or open-circuiting the $i$th mode in this case will make no difference and a zero and a pole of $X_i^{(\mathrm{in})}$ will lie very near to each other.

A similar discussion holds for considering modes of the line section at the other short circuit as well as looking into the structure at an intermediate line section. In the latter case, the resonance condition reads

$$X_i^{(\mathrm{in})}\Big|_{\mathrm{L.H.S.}} + X_i^{(\mathrm{in})}\Big|_{\mathrm{R.H.S.}} = 0 \tag{6}$$

where $X_i^{(\mathrm{in})}\Big|_{\mathrm{L.H.S.}}$ and $X_i^{(\mathrm{in})}\Big|_{\mathrm{R.H.S.}}$ is the input reactance of the $i$th mode looking into the L.H.S. and R.H.S. of the structure, respectively.

Let us now consider the resonance frequencies up to a maximum frequency $f_{\max}$ which corresponds to the prescribed resolution of the problem. The different resonances belonging to this frequency band can be classified into two groups. The resonances of the first group can be easily obtained as zeros of most of the different reactance functions $X_i^{(\mathrm{in})}$. They are well isolated from other zeros or poles. The field expansion corresponding to these modes is characterized by a more or less balanced contribution of all modes ($i = 1, 2, \ldots, M$) which are either propagating or evanescent with an appreciate amplitude at the considered end plane. The second group grasps those resonances corresponding to strongly uneven modal distributions. Some of the considered waveguide modes predominate the others. The resonances of this group are easily found as zeros of the reactance functions corresponding to the predominating waveguide modes. The corresponding zeros of the reactance functions describing the weakly excited modes are hardly localized due to the presence of poles in their vicinity. In order to localize all zeros in the prescribed frequency range, the zeros of all reactance functions $X_i^{(\mathrm{in})}$; $i = 1, 2, \ldots, M$ must be localized and compared to each other. Resonances characterizing the first class are repeated almost $M$-times. Other resonances are less repeated or even present in one single $X_i^{(\mathrm{in})}$ only.

## NUMERICAL RESULTS

The present method has been applied to compute the azimuthally independent resonance modes and the irrotational magnetic eigenfunctions of the short-circuited gyrotron cavity shown in Fig. 1. For an accurate modal expansion of the electromagnetic field inside the cavity some hundreds of these modes are required ([15]-[17]). In order to obtain a clear picture of the structure we have considered the input impedance at the R.H.S. end of the cavity because the largest number of propagating waveguide modes inside the cavity is found there.

For the actual determination of the eigenvalues a searching strategy has been developed which has been proved to be numerically efficient and robust. It alternately localizes the zeros and the poles of the input reactance $X_i^{(\mathrm{in})}$. In order to explain the procedure let us assume that the initial wavenumber $k$ is located between the $q$th pole $p_q$ and the $q$th zero $z_q$ of $X_i^{(\mathrm{in})}$ ($p_q < k < z_q$). In this case, Newton's procedure is used to compute $z_q$ which works very well for a reactance function. Then, Newton's procedure is applied again to the input susceptance $B_i^{(\mathrm{in})} = -1/X_i^{(\mathrm{in})}$ (the zeros of which are the poles of the input reactance and vice versa) starting from $z_q + \varepsilon$ in order to localize the $(q+1)$th pole $p_{q+1}$ of $X_i^{(\mathrm{in})}$. After this has been done, the procedure starts from the beginning with an initial value $p_{q+1} + \varepsilon$ for $k$. In this way the zeros of $X_i^{(\mathrm{in})}$ are successively obtained. If the $i$th waveguide mode is evanescent or weakly excited at the R.H.S. end of the cavity Newton's procedure may skip over a zero and a pole which are very close to each other. Hence, the search strategy should be applied to at least all propagating modes. The reliability of the procedure can be further improved

if more than one cross section of the cavity is considered. The increment $\varepsilon$ should not be confused with a conventional search step because the only purpose of this quantity is to ensure that Newton's procedure is started at $k > p_q$ ($k > z_q$) after the $q$th pole (zero) has been localized. Therefore, $\varepsilon$ should be made as small as possible in order to avoid that any eigenvalues are overlooked.

In Tables 1 and 2 numerical results corresponding to resonance modes and irrotational magnetic eigenfunctions, respectively, are given. A + (-) indicates that a solution is (is not) detected in the corresponding waveguide mode.

For both resonance modes and the irrotational magnetic eigenfunctions the first eigenvalues are found in the dominant waveguide mode only because all other waveguide modes are evanescent at these frequencies. Note that in case of the irrotational magnetic eigenfunctions the $TE_{00}$ waveguide mode has the lowest cutoff wavenumber ($k_c = 0$). This mode is characterized by a transversely constant axial magnetic field only and is not excited if a resonance mode is considered.

The resonance mode #91 and the irrotational magnetic eigenfunction #108 belong to the group of resonances in which some waveguide modes predominate others since the corresponding eigenvalues are only found in the reactance function corresponding to one and two waveguide modes, respectively. Note that although the eigenvalues #367 and #368 corresponding to the irrotational magnetic eigenfunctions are very close to each other they are present in the reactance functions corresponding to a number of waveguide modes. On the other hand, the irrotational magnetic eigenfunction #369, which has been detected with a conventional search strategy, does not appear in any reactance function at the R.H.S. end of the cavity. This solution is however present in the reactance function #5 at the L.H.S. end of the cavity which demonstrates that it is important to look at the input impedances at more than one single cross section of the cavity.

## CONCLUSIONS

It has been demonstrated that it is always possible to find a reactance (or susceptance) function having the same set of zeros as the original determinental equation which determines the eigenvalues of resonance and irrotational cavity modes as well as waveguide modes. It has been shown how use of Foster's theorem can be made in order to systematically determine the zeros of such functions. Numerical results have been given for the azimuthally independent resonance modes and the irrotational magnetic eigenfunctions of a short-circuited gyrotron cavity.

## ACKNOWLEDGEMENT

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## REFERENCES

[1] *A. Wexler, "Solution of waveguide discontinuities by modal analysis," IEEE Trans. Microwave Theory Tech., vol. MTT-15, pp. 508–517, 1967.*

[2] *J. Strube and F. Arndt, "Rigorous hybrid-mode analysis of the transition from rectangular to shielded dielectric image guide," IEEE Trans. Microwave Theory Tech., vol. MTT-33, pp. 391–401, 1985.*

[3] *A. S. Omar and K. Schünemann, "Complex and backward-wave modes in inhomogeneously filled waveguides," IEEE Trans. Microwave Theory Tech., vol. MTT-35, pp. 268–275, 1987.*

[4] *C. J. Railton and T. Rozzi, "Complex modes in boxed microstrip," IEEE Trans. Microwave Theory Tech., vol. MTT-36, pp. 865–874, 1988.*

[5] *A. S. Omar and K. Schünemann, "Analysis of waveguides with metal inserts," IEEE Trans. Microwave Theory Tech., vol. MTT-37, pp. 1924–1932, 1989.*

[6] *A. S. Omar and K. Schünemann, "Application of the generalized spectral-domain technique to the analysis of rectangular waveguides with rectangular and circular metal inserts," IEEE Trans. Microwave Theory Tech., vol. MTT-39, pp. 944–952, 1991.*

[7] *A. Jöstingmeier, A. S. Omar, C. Rieckmann, and S. Lütgert, "Application of the GSD technique to the analysis of slot-coupled waveguides," IEEE Trans. Microwave Theory Tech., vol. MTT-42, pp. 2139–2148, 1994.*

[8] *A. S. Omar and K. Schünemann, "A new approach to the analysis of inhomogeneously filled cavities," Proc. 18'th EuMC (Stockholm), 1988, pp. 527–533.*

[9] *J. M. Neilson, P. E. Latham, M. Caplan, and W. G. Lawson, "Determination of the resonant frequencies in a complex cavity using the scattering matrix formulation," IEEE Trans. Microwave Theory Tech., vol. MTT-37, pp. 1165–1169, 1989.*

[10] *F. Alessandri, M. Mongiardo, and R. Sorrentino, "Computer-aided design of beam forming networks for modern satellite antennas," IEEE Trans. Microwave Theory Tech., vol. MTT-40, pp. 1117–1127, 1992.*

[11] *A. Jöstingmeier and A. S. Omar, "Analysis of the scattering by dielectric bodies using the SIE formulation," IEEE Trans. Microwave Theory Tech., vol. MTT-42, pp. 471–480, 1994.*

[12] *A. Jöstingmeier, C. Rieckmann, and A. S. Omar, "Rigorous and numerically efficient computation of the irrotational electric and magnetic eigenfunctions of complex gyrotron cavities," Proc. IEEE MTT-S Symp. (San Diego), pp. 321–324, 1994.*

[13] *R. Sorrentino and T. Itoh, "Transverse resonance analysis of finline discontinuities," IEEE Trans. Microwave Theory Tech., vol. MTT-32, pp. 1633–1638, 1984.*

[14] *R. E. Collin, Foundations for Microwave Engineering. New York: Mc Graw-Hill, 1966.*

[15] *A. Jöstingmeier, C. Rieckmann, and A. S. Omar, "Frequency domain analysis of gyrotrons based on complete eigenfunction expansions," Conf. Digest 19'th Int. Conf. on Infrared and Millimeter Waves (Sendai), pp. 345–346, 1994.*

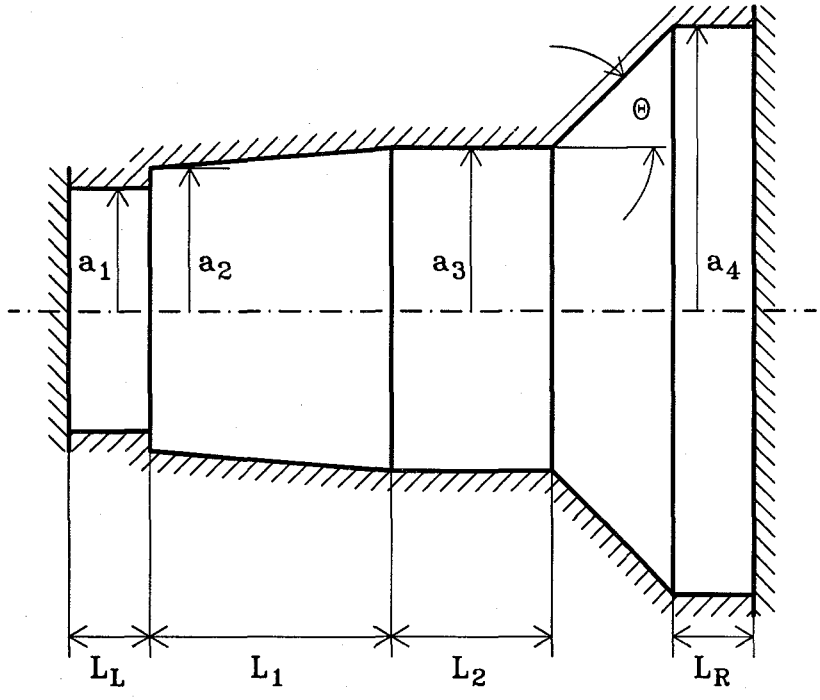


Figure 1: Longitudinal section of a low Q gyrotron cavity.
Parameters: $a_1 = 4.7$ mm, $a_2 = 5.175$ mm, $a_3 = 5.3$ mm, $a_4 = 8.01$ mm,
$\Theta = 36°$, $L_L = 3$ mm, $L_1 = 18.87$ mm, $L_2 = 9.73$ mm, $L_R = 5$ mm.

| # of resonance | wavenumber | # of waveguide mode | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 1 | 0.6123 | + | - | - | - | - | - | - | - | - |
| 2 | 0.7321 | + | - | - | - | - | - | - | - | - |
| 3 | 0.7508 | + | - | - | - | - | - | - | - | - |
| 4 | 0.7755 | + | - | - | - | - | - | - | - | - |
| 5 | 0.8074 | + | - | - | - | - | - | - | - | - |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| 91 | 2.5516 | - | - | + | - | - | - | - | - | - |
| 92 | 2.5593 | + | + | + | - | + | - | - | - | - |
| 93 | 2.5663 | - | + | + | + | + | - | - | - | - |
| 94 | 2.5703 | + | + | + | + | - | - | - | - | - |
| 95 | 2.5785 | + | + | + | + | + | - | - | - | - |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| 251 | 4.0072 | - | - | + | + | + | - | + | - | + |
| 252 | 4.0100 | + | + | + | - | + | + | + | - | - |
| 253 | 4.0148 | + | + | + | + | + | + | + | - | + |
| 254 | 4.0228 | + | + | + | + | + | + | - | + | + |
| 254 | 4.0312 | - | - | + | + | + | + | + | + | + |

Table 1: Map of zeros corresponding to the individual waveguide modes for a resonant cavity mode.

| # of resonance | wavenumber | # of waveguide mode | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 1 | 0.0694 | + | - | - | - | - | - | - | - | - | - | - |
| 2 | 0.1494 | + | - | - | - | - | - | - | - | - | - | - |
| 3 | 0.2341 | + | - | - | - | - | - | - | - | - | - | - |
| 4 | 0.3166 | + | - | - | - | - | - | - | - | - | - | - |
| 5 | 0.3935 | + | - | - | - | - | - | - | - | - | - | - |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| 105 | 2.2647 | + | + | + | + | + | + | - | - | - | - | - |
| 106 | 2.2807 | + | + | + | + | + | + | - | - | - | - | - |
| 107 | 2.2900 | + | - | + | + | + | + | - | - | - | - | - |
| 108 | 2.2973 | + | + | - | - | - | - | - | - | - | - | - |
| 109 | 2.3139 | - | + | + | + | + | + | - | - | - | - | - |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| 366 | 4.3602 | + | + | + | - | - | + | + | + | - | - | - |
| 367 | 4.3655 | + | + | - | - | + | - | - | + | + | + | - |
| 368 | 4.3659 | - | - | + | - | - | - | - | + | + | - | + |
| 369 | 4.3724 | - | - | - | - | - | - | - | - | - | - | - |
| 370 | 4.3790 | - | - | - | - | + | - | + | - | + | + | + |

Table 2: Map of zeros corresponding to the individual waveguide modes for an irrotational magnetic cavity eigenfunction.